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Prolongation structures of the supersymmetric sine-Gordon equation and infinite-dimensional superalgebras

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Abstract. Prolongation structures of the supersymmetric sine-Gordon equation are discussed. It will be shown that an infinite-dimensional superalgebra is associated with these structures and that a linear representation of the algebra gives the super Lax pairs of the equation.

1. Introduction

It is well known that completely integrable non-linear equations in two-dimensional spacetime have common features. They have Bäcklund transformations and an infinite number of conserved quantities. The standard approach to show the existence of such properties is the inverse scattering method. This method consists of a pair of linear auxiliary equations (the Lax pair), the integrability condition of which gives the non-linear equations.

The prolongation method proposed by Estabrook and Wahlquist [1] has been shown to give a systematic way of finding the Lax pairs [2-6]. Recently we have shown that the prolongation method reveals the existence of infinite-dimensional algebras (such as the Kac-Moody algebra and the Virasoro algebra) and associated non-linear equations (the sine-Gordon equation and the Ernst equation) [7]. It was shown also that linear representations of these algebras give the Lax pair of the equations.

In this paper we will discuss prolongation structures of the non-linear equation, which includes fermion fields, and show that the prolongation method plays an important role in finding the Lax pairs of the equation.

A standard way of incorporating fermion fields retaining complete integrability is the supersymmetric extension of the completely integrable bosonic system. An example of such an extension is the supersymmetric sine-Gordon equation. It was shown that this equation has an infinite number of conserved quantities [8] and the Lax pair [9-11].

In the classical theory fermion fields have to be treated as odd elements of the Grassmann algebra. Then this model has fields with both characters of the Grassmann algebra (odd and even elements). This fact suggests that there are scattering parameters and pseudopotentials of both odd and even characters and that there is an infinite-dimensional superalgebra associated with this model.

In the next section we will obtain differential equations which have to be satisfied when we prolong a differential system of the supersymmetric sine-Gordon equation. In § 3 these equations are discussed algebraically in terms of vector fields on a supermanifold and an infinite-dimensional superalgebra will be defined. In § 4 we will

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show that a linear representation of the algebra provides the super Lax pairs of the supersymmetric sine-Gordon equation.

2. Prolongation equations of the supersymmetric sine-Gordon model

The supersymmetric sine-Gordon equation in two-dimensional spacetime is given by

$$
\begin{aligned}
&\partial_\xi \partial_\eta \phi = \sin\phi - 2\mathrm{i}\psi_1\psi_2 \sin(\phi/2) \\
&\partial_\eta \psi_1 + \cos(\phi/2)\psi_2 = 0 \\
&\partial_\xi \psi_2 + \cos(\phi/2)\psi_1 = 0
\end{aligned}
\tag{2.1}
$$

where ψ_1 and ψ_2 are odd elements of the Grassmann algebra. Then they have the following properties

$$
\begin{aligned}
&\psi_1(x)\psi_1(x) = 0 \qquad \psi_2(x)\psi_2(x) = 0 \\
&\psi_1(x)\psi_2(x') + \psi_2(x')\psi_1(x) = 0.
\end{aligned}
\tag{2.2}
$$

These field equations are rewritten in terms of 2-forms α_i and β_i $(i = 1, 2)$ on a six-dimensional supermanifold with coordinates $\{\phi, \pi, \psi_1, \psi_2, \eta, \xi\}$ as

$$
\begin{aligned}
&\alpha_1 = \mathrm{d}\phi \wedge \mathrm{d}\xi - \pi\, \mathrm{d}\eta \wedge \mathrm{d}\xi \\
&\alpha_2 = \mathrm{d}\pi \wedge \mathrm{d}\eta + (\sin\phi - 2\mathrm{i}\psi_1\psi_2)\, \mathrm{d}\eta \wedge \mathrm{d}\xi \\
&\beta_1 = \mathrm{d}\psi_2 \wedge \mathrm{d}\xi + \cos(\phi/2)\psi_1\, \mathrm{d}\eta \wedge \mathrm{d}\xi \\
&\beta_2 = \mathrm{d}\psi_1 \wedge \mathrm{d}\eta - \cos(\phi/2)\psi_2\, \mathrm{d}\eta \wedge \mathrm{d}\xi.
\end{aligned}
\tag{2.3}
$$

As we can easily show, the differential system $\{\alpha_i, \beta_i\}$ is closed

$$
\mathrm{d}\alpha_i \in I(\alpha, \beta) \qquad \mathrm{d}\beta_i \in I(\alpha, \beta)
\tag{2.4}
$$

where $I(\alpha, \beta)$ denotes an ideal generated by the set $\{\alpha_i, \beta_i\}$.

Now we will consider the prolongation of the differential system $\{\alpha, \beta\}$ by adding 1-forms Q^i and Ω^μ $(i = 1, 2, \ldots r;\ \mu = 1, 2, \ldots, s)$

$$
\begin{aligned}
&Q^i = -\mathrm{d}q^i + F^i(\pi, \phi, \psi, q, \omega)\, \mathrm{d}\eta + G^i(\pi, \phi, \psi, q, \omega)\, \mathrm{d}\xi \\
&\Omega^\mu = -\mathrm{d}\omega^\mu + \theta^\mu(\pi, \phi, \psi, q, \omega)\, \mathrm{d}\eta + \Sigma^\mu(\pi, \phi, \psi, q, \omega)\, \mathrm{d}\xi
\end{aligned}
\tag{2.5}
$$

where r and s are determined later. In (2.5) q^i and ω^μ are called pseudopotentials. It is assumed that q^i and ω^μ are even and odd elements, respectively, of the Grassmann algebra. Then Q^i, F^i and G^i are even elements and Ω^μ, θ^μ and Σ^μ are odd elements.

The prolonged differential system $\{\alpha, \beta, Q, \Omega\}$ is generated on a $(r+4, s+2)$-dimensional supermanifold of $\{\phi, \pi, \psi_1, \psi_2, q, \omega, \eta, \xi\}$. This differential system is not in general closed. Then the prolongation can be carried out, provided that Q^i and Ω^μ satisfy the integrability conditions

$$
\mathrm{d}Q^i \in I(\alpha, \beta, Q, \Omega) \tag{2.6a}
$$

$$
\mathrm{d}\Omega^\mu \in I(\alpha, \beta, Q, \Omega). \tag{2.6b}
$$

From (2.6a) we have differential equations of F^i and G^i:

$$
\partial_\phi F^i = 0 \qquad \partial_\pi G^i = 0 \qquad \partial_{\psi_2} F^i = 0 \qquad \partial_{\psi_1} G^i = 0 \tag{2.7a}
$$

and

$$-G^j\partial_j F^i + F^j\partial_j G^i - \Sigma^\mu\partial_\mu F^i + \theta^\mu\partial_\mu G^i - [\sin\phi - 2\mathrm{i}\psi_1\psi_2\sin(\phi/2)]\partial_\pi F^i$$
$$+\cos(\phi/2)\psi_2\partial_{\psi_1}F^i + \pi\partial_\phi G^i - \cos(\phi/2)\psi_1\partial_{\psi_2}G^i = 0 \qquad (2.7b)$$

where $\partial_i = \partial/\partial q^i$ and $\partial_\mu = \partial_L/\partial\omega^\mu$ (the left derivative with ω^μ).

Equation (2.6b) gives differential equations of θ^μ and Ω^μ:

$$\partial_\phi\theta^\mu = 0 \qquad \partial_{\psi_2}\theta^\mu = 0 \qquad \partial_\pi\Sigma^\mu = 0 \qquad \partial_{\psi_1}\Sigma^\mu = 0 \qquad (2.8a)$$

and

$$-G^i\partial_i\theta^\mu + F^i\partial_i\Sigma^\mu - \Sigma^\nu\partial_\nu\theta^\mu + \theta^\nu\partial_\nu\Sigma^\mu - [\sin\phi - 2\mathrm{i}\psi_1\psi_2\sin(\phi/2)]\partial_\pi\theta^\mu$$
$$+\cos(\phi/2)\psi_2\partial\psi_1\theta^\mu + \pi\partial_\phi\Sigma^\mu - \cos(\phi/2)\psi_1\partial_{\psi_2}\Sigma^\mu = 0. \qquad (2.8b)$$

Equation (2.7a) and (2.8a) mean that F^i and θ^μ are functions of π, ψ_1, q and ω and that G^i and Σ^μ are functions of ϕ, ψ_2, q and ω:

$$F^i = F^i(\pi, \psi_1, q, \omega) \qquad G^i = G^i(\phi, \psi_2, q, \omega)$$
$$\theta^\mu = \theta^\mu(\pi, \psi_1, q, \omega) \qquad \Sigma^\mu = \Sigma^\mu(\phi, \psi_2, q, \omega). \qquad (2.9)$$

Since $(\psi_1)^2 = 0$ and $(\psi_2)^2 = 0$, we can represent F, G, θ and Σ as

$$F^i = F_0^i(\pi, q, \omega) + F_1^i(\pi, q, \omega)\psi_1$$
$$G^i = G_0^i(\phi, q, \omega) + G_2^i(\phi, q, \omega)\psi_2$$
$$\theta^\mu = \theta_0^\mu(\pi, q, \omega) + \theta_1^\mu(\pi, q, \omega)\psi_1$$
$$\Sigma^\mu = \Sigma_0^\mu(\phi, q, \omega) + \Sigma_2^\mu(\phi, q, \omega)\psi_2 \qquad (2.10)$$

where F_0^i, G_0^i, θ_1^μ and Σ_2^μ are even elements and F_1^i, G_2^i, θ_0^μ and Σ_0^μ are odd elements.

By introducing (2.10) into (2.7b) and (2.8b) we obtain the following differential equations:

$$(F_0^j\partial_j + \theta_0^\mu\partial_\mu)G_0^i - (G_0^j\partial_j + \Sigma_0^\mu\partial_\mu)F_0^i - \sin\phi\partial_\pi F_0^i + \pi\partial_\phi G_0^i = 0$$
$$(F_1^j\partial_j - \theta_1^\mu\partial_\mu)G_0^i - (G_0^j\partial_j + \Sigma_0^\mu\partial_\mu)F_1^i - \sin\phi\partial_\pi F_1^i - \cos(\phi/2)G_2^i = 0$$
$$(F_0^j\partial_j + \theta_0^\mu\partial_\mu)G_2^i - (G_2^j\partial_j - \Sigma_2^\mu\partial_\mu)F_0^i + \pi\partial_\phi G_2^i + \cos(\phi/2)F_1^i = 0$$
$$(F_1^j\partial_j - \theta_1^\mu\partial_\mu)G_2^i + (G_2^j\partial_j - \Sigma_2^\mu\partial_\mu)F_1^i + 2\mathrm{i}\sin(\phi/2)\partial_\pi F_0^i = 0 \qquad (2.11)$$

and

$$(F_0^i\partial_i + \theta_0^\nu\partial_\nu)\Sigma_0^\mu - (G_0^i\partial_i + \Sigma_0^\nu\partial_\mu)\theta_0^\mu - \sin\phi\partial_\pi\theta_0^\mu + \pi\partial_\phi\Sigma_0^\mu = 0$$
$$(F_1^i\partial_i - \theta_1^\nu\partial_\nu)\Sigma_0^\mu + (G_0^i\partial_i + \Sigma_0^\nu\partial_\nu)\theta_1^\mu + \sin\phi\partial_\pi\theta_1^\mu + \cos(\phi/2)\Sigma_2^\mu = 0$$
$$(F_0^i\partial_i + \theta_0^\nu\partial_\nu)\Sigma_2^\mu + (G_2^i\partial_i - \Sigma_2^\nu\partial_\nu)\theta_0^\mu + \cos(\phi/2)\theta_1^\mu + \pi\partial_\phi\Sigma_2^\mu = 0$$
$$(F_1^i\partial_i - \theta_1^\nu\partial_\nu)\Sigma_2^\mu + (G_2^i\partial_i - \Sigma_2^\nu\partial_\nu)\theta_1^\mu + 2\mathrm{i}\sin(\phi/2)\partial_\pi\theta_0^\mu = 0. \qquad (2.12)$$

In order to prolong the differential system we have to integrate these differential equations. This problem will be solved algebraically in the next two sections.

3. The prolongation algebra

Let us consider a supermanifold with local coordinates $\{q^i, \omega^\mu\}$. Vector fields A, B, $\hat{A}$ and $\hat{B}$ on this manifold are defined by

$$
\begin{aligned}
A &= A^i \frac{\partial}{\partial q^i} + A^\mu \frac{\partial}{\partial \omega^\mu} \qquad & B &= B^i \frac{\partial}{\partial q^i} + B^\mu \frac{\partial}{\partial \omega^\mu} \\
\hat{A} &= \hat{A}^i \frac{\partial}{\partial q^i} + \hat{A}^\mu \frac{\partial}{\partial \omega^\mu} \qquad & \hat{B} &= \hat{B}^i \frac{\partial}{\partial q^i} + \hat{B}^\mu \frac{\partial}{\partial \omega^\mu}
\end{aligned}
\tag{3.1}
$$

where A and B are even elements and $\hat{A}$ and $\hat{B}$ are odd elements.

These vector fields have the Lie brackets which are defined by

$$
\begin{aligned}
[A, B] &= [(A^i\partial_i + A^\nu\partial_\nu)B^j - (B^i\partial_i + B^\nu\partial_\nu)A^j]\frac{\partial}{\partial q^j} \\
&\quad + [(A^i\partial_i + A^\nu\partial_\nu)B^\mu - (B^i\partial_i + B^\nu\partial_\nu)A^\mu]\frac{\partial}{\partial \omega^\mu} \\
[A, \hat{B}] &= [(A^i\partial_i + A^\nu\partial_\nu)\hat{B}^j - (\hat{B}^i\partial_i + \hat{B}^\nu\partial_\nu)A^j]\frac{\partial}{\partial q^j} \\
&\quad + [(A^i\partial_i + A^\nu\partial_\nu)\hat{B}^\mu - (B^i\partial_i + \hat{B}^\nu\partial_\nu)A^\mu]\frac{\partial}{\partial \omega^\mu} \\
[\hat{A}, \hat{B}]_+ &= [(\hat{A}^i\partial_i + \hat{A}^\nu\partial_\nu)\hat{B}^j + (\hat{B}^i\partial_i + \hat{B}^\nu\partial_\nu)\hat{A}^j]\frac{\partial}{\partial q^j} \\
&\quad + [(\hat{A}^i\partial_i + \hat{A}^\nu\partial_\nu)\hat{B}^\mu + (\hat{B}^i\partial_i + \hat{B}^\nu\partial_\nu)\hat{A}^\mu]\frac{\partial}{\partial \omega^\mu}
\end{aligned}
\tag{3.2}
$$

and $[A, \hat{B}] = -[\hat{B}, A]$.

Here we will choose vector fields A, B, $\hat{A}$ and $\hat{B}$ so that components of these vector fields are given in terms of F, G, θ and Σ as

$$
\begin{aligned}
A &= F_0^i \frac{\partial}{\partial q^i} + \theta_0^\mu \frac{\partial}{\partial \omega^\mu} \qquad & B &= G_0^i \frac{\partial}{\partial q^i} + \Sigma_0^\mu \frac{\partial}{\partial \omega^\mu} \\
\hat{A} &= F_1^i \frac{\partial}{\partial q^i} - \theta_1^\mu \frac{\partial}{\partial \omega^\mu} \qquad & \hat{B} &= G_2^i \frac{\partial}{\partial q^i} - \Sigma_2^\mu \frac{\partial}{\partial \omega^\mu}.
\end{aligned}
\tag{3.3}
$$

Then from (2.11), (2.12) and (3.2) we find that they satisfy the Lie brackets:

$$
\begin{aligned}
[A, B] &= \sin\phi\,\partial_\pi A - \pi\partial_\phi B \qquad & [A, \hat{B}] &= -\cos(\phi/2)\hat{A} - \pi\partial_\phi\hat{B} \\
[\hat{A}, B] &= \sin\phi\,\partial_\pi\hat{A} + \cos(\phi/2)\hat{B} \qquad & [\hat{A}, \hat{B}]_+ &= -2\mathrm{i}\sin(\phi/2)\partial_\pi A
\end{aligned}
\tag{3.4}
$$

where $\partial_\pi A$, $\partial_\phi\hat{B}$ and $\partial_\pi\hat{A}$ are defined by

$$
\partial_\pi A = (\partial_\pi A^i)\frac{\partial}{\partial q^i} + (\partial_\pi A^\mu)\frac{\partial}{\partial \omega^\mu} \qquad \partial_\phi\hat{B} = (\partial_\phi B^i)\frac{\partial}{\partial q^i} + (\partial_\phi B^\mu)\frac{\partial}{\partial \omega^\mu}
$$

$$
\partial_\pi\hat{A} = (\partial_\pi\hat{A}^i)\frac{\partial}{\partial q^i} + (\partial_\pi\hat{A}^\mu)\frac{\partial}{\partial \omega^\mu}.
$$

From (3.4) we can show that these vector fields have the following forms:

$$\begin{aligned} &A = X_0 + \pi X_1 && B = Y_0 \sin\phi + Y_1 \cos\phi + Y_2 \\ &\hat{A} = \hat{X}_0 && \hat{B} = \hat{Y}_0 \sin(\phi/2) + \hat{Y}_1 \cos(\phi/2) \end{aligned} \tag{3.5}$$

where $X_i (i = 0, 1)$, $Y_i (i = 0, 1, 2)$, $\hat{X}_0$ and $\hat{Y}_i$ $(i = 0, 1)$ are vector fields which are independent of field variables π and ϕ. In (3.5) X_i, Y_i are even elements and $\hat{X}_0$, $\hat{Y}_i$ are odd elements and they satisfy the Lie brackets:

$$\begin{aligned} &[X_0, Y_0] = X_1 && [X_0, Y_1] = 0 && [X_0, Y_2] = 0 \\ &[X_1, Y_0] = Y_1 && [X_1, Y_1] = -Y_0 && [X_1, Y_2] = 0 \\ &[X_0, \hat{Y}_0] = 0 && [X_0, \hat{Y}_1] = -\hat{X}_0 && [X_1, \hat{Y}_0] = \tfrac{1}{2}\hat{Y}_1 \\ &[X_1, \hat{Y}_1] = -\tfrac{1}{2}\hat{Y}_0, && [\hat{X}_0, Y_0] = \tfrac{1}{2}\hat{Y}_0 && [\hat{X}_0, Y_1] = \tfrac{1}{2}\hat{Y}_1 \\ &[\hat{X}_0, Y_2] = \tfrac{1}{2}\hat{Y}_1 && [\hat{X}_0, \hat{Y}_0]_+ = -2\mathrm{i}X_1 && [\hat{X}_0, \hat{Y}_1]_+ = 0. \end{aligned} \tag{3.6}$$

Here we have to notice that the subset of vector fields $\{X_0, X_1, Y_0, Y_1\}$ satisfies the same prolongation algebra of the sine-Gordon equation.

Thus the differential equations (2.11) and (2.12) are replaced by the incomplete algebra (3.6). Then it is necessary to find the representation of the algebra in spite of solving (2.11) and (2.12).

Next we will consider a $(r+s)$-fold infinite-dimensional manifold with coordinates $\{q_i^{(n)}; i = 1, 2, \ldots, r: \omega_\mu^{(n)}; \mu = 1, 2, \ldots, s\}$, where n runs from $-\infty$ to $+\infty$, and define an infinite number of vector fields $A_{ij}^{(m)}$, $B_{\mu\nu}^{(m)}$, $\Gamma_{\mu i}^{(m)}$ and $\Gamma_{i\mu}^{(m)}$ $(m = 0, \pm 1, \pm 2, \ldots \pm\infty)$ by

$$\begin{aligned} &A_{ij}^{(m)} = \sum_{n=-\infty}^{\infty} q_i^{(n+m)} \frac{\partial}{\partial q_j^{(n)}} && B_{\mu\nu}^{(m)} = \sum_{n=-\infty}^{\infty} \omega_\mu^{(n+m)} \frac{\partial}{\partial \omega_\nu^{(n+m)}} \\ &\Gamma_{\mu i}^{(m)} = \sum_{n=-\infty}^{\infty} \omega_\mu^{(n+m)} \frac{\partial}{\partial q_i^{(m)}} && \Gamma_{i\mu}^{(m)} = \sum_{n=-\infty}^{\infty} q_i^{(n+m)} \frac{\partial}{\partial \omega_\mu^{(n)}} \end{aligned} \tag{3.7}$$

where $A_{ij}^{(m)}$ and $B_{\mu\nu}^{(m)}$ are even elements and $\Gamma_{\mu i}^{(m)}$ and $\Gamma_{i\mu}^{(m)}$ are odd elements.

From (3.2) and (3.7) we can show that these four kinds of vector fields satisfy an infinite-dimensional superalgebra given by

$$\begin{aligned} &[A_{ij}^{(m)}, A_{kl}^{(n)}] = \delta_{jk} A_{il}^{(m+n)} - \delta_{il} A_{kj}^{(m+n)} \\ &[B_{\mu\nu}^{(m)}, B_{\lambda\sigma}^{(n)}] = \delta_{\lambda\nu} B_{\mu\sigma}^{(m+n)} - \delta_{\mu\sigma} B_{\lambda\nu}^{(m+n)} \\ &[A_{ij}^{(m)}, B_{\mu\nu}^{(n)}] = 0 \\ &[A_{ij}^{(m)}, \Gamma_{\mu k}^{(n)}] = -\delta_{ik} \Gamma_{\mu j}^{(m+n)} \qquad [A_{ij}^{(m)}, \Gamma_{k\mu}^{(n)}] = \delta_{jk} \Gamma_{i\mu}^{(m+n)} \\ &[B_{\mu\nu}^{(m)}, \Gamma_{\lambda i}^{(n)}] = \delta_{\nu\lambda} \Gamma_{\mu i}^{(m+n)} \qquad [B_{\mu\nu}^{(m)}, \Gamma_{i\lambda}^{(n)}] = -\delta_{\mu\lambda} \Gamma_{i\nu}^{(m+n)} \\ &[\Gamma_{\mu i}^{(m)}, \Gamma_{\nu j}^{(n)}]_+ = 0 \qquad [\Gamma_{i\mu}^{(m)}, \Gamma_{j\nu}^{(n)}]_+ = 0 \\ &[\Gamma_{\mu i}^{(m)}, \Gamma_{j\nu}^{(n)}]_+ = \delta_{ij} B_{\mu\nu}^{(m+n)} + \delta_{\mu\nu} A_{ji}^{(m+n)}. \end{aligned} \tag{3.8}$$

In the following we restrict our considerations to a special case with $r = s = 2$, and discuss a subset of vector fields $\{M_i^{(m)}, i = 1, 2, 3;\ N^{(m)}, \Lambda_\mu^{(m)}, \mu = 0, 1, 2, 3\}$ defined by some linear combinations of $A_{ij}^{(m)}$, $B_{\mu\nu}^{(m)}$, $\Gamma_{\mu i}^{(m)}$ and $\Gamma_{i\mu}^{(m)}$:

$$M_i^{(m)} = \frac{1}{2} \sum_{n=-\infty}^{\infty} q_a^{(n+m)} (\sigma_i)_{ab} \frac{\partial}{\partial q_b^{(n)}} \qquad a, b = 1, 2$$

$$
\begin{aligned}
N^{(m)} &= \frac{1}{2} \sum_{n=-\infty}^{\infty} \omega_a^{(n+m)} (\sigma_3)_{ab} \frac{\partial}{\partial \omega_b^{(n)}} \\
\Lambda_0^{(m)} &= \frac{1}{\sqrt{2}} \sum_{n=-\infty}^{\infty} \left(q_a^{(n+m)} (\sigma_3)_{ab} \frac{\partial}{\partial \omega_b^{(n)}} + \omega_a^{(n+m)} (\sigma_0)_{ab} \frac{\partial}{\partial q_b^{(n)}} \right) \\
\Lambda_1^{(m)} &= \frac{1}{\sqrt{2}} \sum_{n=-\infty}^{\infty} \left(q_a^{(n+m)} (\sigma_1)_{ab} \frac{\partial}{\partial \omega_b^{(n)}} - \omega_a^{(n+m)} (\sigma_2)_{ab} \frac{\partial}{\partial q_b^{(n)}} \right) \\
\Lambda_2^{(m)} &= \frac{1}{\sqrt{2}} \sum_{n=-\infty}^{\infty} \left(q_a^{(n+m)} (\sigma_0)_{ab} \frac{\partial}{\partial \omega_b^{(n)}} + \omega_a^{(n+m)} (\sigma_3)_{ab} \frac{\partial}{\partial q_b^{(n)}} \right) \\
\Lambda_3^{(m)} &= \frac{1}{\sqrt{2}} \sum_{n=-\infty}^{\infty} \left(q_a^{(n+m)} (\sigma_2)_{ab} \frac{\partial}{\partial \omega_b^{(n)}} + \omega_a^{(n+m)} (\sigma_1)_{ab} \frac{\partial}{\partial q_b^{(n)}} \right)
\end{aligned}
\tag{3.9}
$$

where σ_i is the Pauli matrix and $-\mathrm{i}\sigma_0$ is the unit matrix.

It can be shown that this set of vector fields makes an infinite-dimensional supersubalgebra

$$
\begin{aligned}
&[M_i^{(m)}, M_j^{(n)}] = \mathrm{i}\varepsilon_{ijk} M_k^{(m+n)} \qquad && [M_i^{(m)}, N^{(n)}] = 0 \\
&[M_i^{(m)}, \Lambda_\mu^{(n)}] = \tfrac{1}{2}\mathrm{i} f_{i\mu\nu} \Lambda_\nu^{(m+n)} \qquad && [N^{(m)}, \Lambda_\mu^{(n)}] = \tfrac{1}{2}\mathrm{i} g_{\mu\nu} \Lambda_\nu^{(m+n)} \\
&[\Lambda_\mu^{(m)}, \Lambda_\nu^{(n)}]_+ = 2\mathrm{i} d_{i\mu\nu} M_i^{(m+n)} + 2\mathrm{i}\delta_{\mu\nu} N^{(m+n)}
\end{aligned}
\tag{3.10}
$$

where $f_{i\mu\nu}$, $g_{\mu\nu}$ and $d_{i\mu\nu}$ are represented in terms of 4×4 matrices $f_i = \{f_{i\mu\nu}\}$, $g = \{g_{\mu\nu}\}$, $d_i = \{d_{i\mu\nu}\}$:

$$
\begin{aligned}
&f_1 = \begin{pmatrix} 0 & -\sigma_1 \\ \sigma_1 & 0 \end{pmatrix} \qquad f_2 = \begin{pmatrix} \mathrm{i}\sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix} \qquad f_3 = \begin{pmatrix} 0 & -\sigma_3 \\ \sigma_3 & 0 \end{pmatrix} \\
&g = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad d_1 = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix} \qquad d_2 = \begin{pmatrix} 0 & \mathrm{i}\sigma_2 \\ -\mathrm{i}\sigma_2 & 0 \end{pmatrix} \qquad d_3 = \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}.
\end{aligned}
\tag{3.11}
$$

In the next section we will show that (3.10) is the characteristic algebra associated with the supersymmetric sine-Gordon equation.

4. The representation of the prolongation algebra and the super Lax pairs

We will return to find the representation of the incomplete prolongation algebra (3.6) and to complete the prolongation structures of the supersymmetric sine-Gordon equation.

We will identify elements of the prolongation algebra (3.6) X_i, Y_i, $\hat{X}_0$ and $\hat{Y}_i$ with some elements of the infinite-dimensional superalgebra (3.10) as

$$
\begin{aligned}
&X_0 = M_3^{(2)} + N^{(2)} \qquad && X_1 = M_2^{(0)} \qquad && Y_0 = -M_1^{(-2)} \qquad && Y_1 = M_3^{(-2)} \\
&Y_2 = N^{(-2)} \qquad && \hat{X}_0 = \mathrm{i}\Lambda_1^{(1)} \qquad && \hat{Y}_0 = \Lambda_2^{(-1)} \qquad && \hat{Y}_1 = \Lambda_3^{(-1)}.
\end{aligned}
\tag{4.1}
$$

Then we can easily show that all the Lie brackets of (3.6) are satisfied. This fact means that the prolongation algebra can be embedded in the algebra (3.10).

From (3.9) and (4.1) we can obtain the explicit representation of vector fields for the prolongation algebra

$$
\begin{aligned}
X_0&=\frac{1}{2}\sum_{n=-\infty}^{\infty}\left(q_1^{(n+2)}\frac{\partial}{\partial q_1^{(n)}}-q_2^{(n+2)}\frac{\partial}{\partial q_2^{(n)}}\right)+\frac{1}{2}\sum_{n=-\infty}^{\infty}\left(\omega_1^{(n+2)}\frac{\partial}{\partial \omega_1^{(n)}}-\omega_2^{(n+2)}\frac{\partial}{\partial \omega_2^{(n)}}\right)\\
X_1&=\frac{1}{2}\sum_{n=-\infty}^{\infty}\left(q_2^{(n)}\frac{\partial}{\partial q_1^{(n)}}-q_1^{(n)}\frac{\partial}{\partial q_2^{(n)}}\right)\\
Y_0&=\frac{-1}{2}\sum_{n=-\infty}^{\infty}\left(q_2^{(n-2)}\frac{\partial}{\partial q_1^{(n)}}+q_1^{(n-2)}\frac{\partial}{\partial q_2^{(n)}}\right)\\
Y_1&=\frac{1}{2}\sum_{n=-\infty}^{\infty}\left(q_1^{(n-2)}\frac{\partial}{\partial q_1^{(n)}}-q_2^{(n-2)}\frac{\partial}{\partial q_2^{(n)}}\right)\\
Y_2&=\frac{1}{2}\sum_{n=-\infty}^{\infty}\left(\omega_1^{(n-2)}\frac{\partial}{\partial \omega_1^{(n)}}-\omega_2^{(n-2)}\frac{\partial}{\partial \omega_2^{(n)}}\right)\\
\hat{X}_0&=\frac{1}{\sqrt{2}}\sum_{n=-\infty}^{\infty}\left(\omega_2^{(n+1)}\frac{\partial}{\partial q_1^{(n)}}-\omega_1^{(n+1)}\frac{\partial}{\partial q_2^{(n)}}\right)+\frac{\mathrm{i}}{\sqrt{2}}\sum_{n=-\infty}^{\infty}\left(q_2^{(n+1)}\frac{\partial}{\partial \omega_1^{(n)}}+q_1^{(n+1)}\frac{\partial}{\partial \omega_2^{(n)}}\right)\\
\hat{Y}_0&=\frac{1}{\sqrt{2}}\sum_{n=-\infty}^{\infty}\left(\omega_1^{(n-1)}\frac{\partial}{\partial q_1^{(n)}}-\omega_2^{(n-1)}\frac{\partial}{\partial q_2^{(n)}}\right)+\frac{\mathrm{i}}{\sqrt{2}}\sum_{n=-\infty}^{\infty}\left(q_1^{(n-1)}\frac{\partial}{\partial \omega_1^{(n)}}+q_2^{(n-1)}\frac{\partial}{\partial \omega_2^{(n)}}\right)\\
\hat{Y}_1&=\frac{1}{\sqrt{2}}\sum_{n=-\infty}^{\infty}\left(\omega_2^{(n-1)}\frac{\partial}{\partial q_1^{(n)}}+\omega_1^{(n-1)}\frac{\partial}{\partial q_2^{(n)}}\right)+\frac{\mathrm{i}}{\sqrt{2}}\sum_{n=-\infty}^{\infty}\left(q_2^{(n-1)}\frac{\partial}{\partial \omega_1^{(n)}}-q_1^{(n-1)}\frac{\partial}{\partial \omega_2^{(n)}}\right).
\end{aligned}
\tag{4.2}
$$

Then from (3.3), (3.5) and (4.2) we have solutions of the differential equations (2.11) and (2.12)

$$
\begin{aligned}
&(F_0)_1^n=\tfrac{1}{2}q_1^{(n+2)}+\tfrac{1}{2}\pi q_2^{(n)} \qquad (F_0)_2^n=-\tfrac{1}{2}q_2^{(n+2)}-\tfrac{1}{2}\pi q_1^{(n)}\\
&(G_0)_1^n=-\tfrac{1}{2}\sin\phi q_2^{(n-2)}+\tfrac{1}{2}\cos\phi q_1^{(n-2)} \qquad (G_0)_2^n=-\tfrac{1}{2}\sin\phi q_1^{(n-2)}-\tfrac{1}{2}\cos\phi q_2^{(n-2)}\\
&(\theta_0)_1^n=\tfrac{1}{2}\omega_1^{(n+2)} \qquad (\theta_0)_2^n=-\tfrac{1}{2}\omega_2^{(n+2)}\\
&(\Sigma_0)_1^n=\tfrac{1}{2}\omega_1^{(n-2)} \qquad (\Sigma_0)_2^n=-\tfrac{1}{2}\omega_2^{(n-2)}\\
&(F_1)_1^n=(1/\sqrt{2})\omega_2^{(n+1)} \qquad (F_1)_2^n=-(1/\sqrt{2})\omega_1^{(n+1)}\\
&(G_2)_1^n=(1/\sqrt{2})\sin(\phi/2)\omega_1^{(n-1)}+(1/\sqrt{2})\cos(\phi/2)\omega_2^{(n-1)}\\
&(G_2)_2^n=-(1/\sqrt{2})\sin(\phi/2)\omega_2^{(n-1)}+(1/\sqrt{2})\cos(\phi/2)\omega_1^{(n-1)}\\
&(\theta_1)_1^n=-(\mathrm{i}/\sqrt{2})q_2^{(n+1)} \qquad (\theta_1)_2^n=-(\mathrm{i}/\sqrt{2})q_1^{(n+1)}\\
&(\Sigma_2)_1=-(\mathrm{i}/\sqrt{2})\sin(\phi/2)q_1^{(n-1)}-(\mathrm{i}/\sqrt{2})\cos(\phi/2)q_2^{(n-1)}\\
&(\Sigma_2)_2=-(\mathrm{i}/\sqrt{2})\sin(\phi/2)q_2^{(n-1)}+(\mathrm{i}/\sqrt{2})\cos(\phi/2)q_1^{(n-1)}.
\end{aligned}
\tag{4.3}
$$

Lastly we can show that there are an infinite number of pseudopotentials $q^{(n)}$ (even element) and $\omega_\mu^{(n)}$ (odd element) which satisfy differential equations

$$
\begin{aligned}
\partial_\eta q_1^{(n)}&=\tfrac{1}{2}(q_1^{(n+2)}+\pi q_2^{(n)})+(1/\sqrt{2})\omega_2^{(n+1)}\psi_1\\
\partial_\eta q_2^{(n)}&=-\tfrac{1}{2}(q_2^{(n+2)}+\pi q_1^{(n)})-(1/\sqrt{2})\omega_1^{(n+1)}\psi_1
\end{aligned}
\tag{4.4}
$$

$$
\begin{aligned}
\partial_\xi q_1^{(n)}&=\tfrac{1}{2}(-\sin\phi q_2^{(n-2)}+\cos\phi q_1^{(n-2)})+(1/\sqrt{2})(\sin(\phi/2)\omega_1^{(n-1)}+\cos(\phi/2)\omega_2^{(n-1)})\psi_2\\
\partial_\xi q_2^{(n)}&=-\tfrac{1}{2}(\sin\phi q_1^{(n-2)}+\cos\phi q_2^{(n-2)})+(1/\sqrt{2})[-\sin(\phi/2)\omega_2^{(n-1)}+\cos(\phi/2)\omega_1^{(n-1)}]\psi_2
\end{aligned}
\tag{4.5}
$$

$$\begin{aligned}\partial_\eta\omega_1^{(n)} &= \tfrac{1}{2}\omega_1^{(n+2)} - (\mathrm{i}/\sqrt{2})q_2^{(n+1)}\psi_1 \\ \partial_\eta\omega_2^{(n)} &= -\tfrac{1}{2}\omega_2^{(n+2)} - (\mathrm{i}/\sqrt{2})q_1^{(n+1)}\psi_1\end{aligned} \tag{4.6}$$

$$\begin{aligned}\partial_\xi\omega_1^{(n)} &= \tfrac{1}{2}\omega_1^{(n-2)} - (\mathrm{i}/\sqrt{2})[\sin(\phi/2)q_1^{(n-1)} + \cos(\phi/2)q_2^{(n-1)}]\psi_2 \\ \partial_\xi\omega_2^{(n)} &= -\tfrac{1}{2}\omega_2^{(n-2)} - (\mathrm{i}/\sqrt{2})[\sin(\phi/2)q_2^{(n-1)} - \cos(\phi/2)q_1^{(n-1)}]\psi_2.\end{aligned} \tag{4.7}$$

These results indicate that we can prolong the differential systems $\{\alpha, \beta\}$ on the six-dimensional supermanifold $\{\phi, \pi, \psi_1, \psi_2, \eta, \xi\}$ to the differential system $\{\alpha, \beta, Q, \Omega\}$ on the infinite-dimensional supermanifold $\{\phi, \pi, \psi_1, \psi_2, q_i^{(n)}, \omega_\mu^{(n)}, \eta, \xi\}$.

Next we will introduce a λ-dependent pseudopotential $q_i(\lambda)$, $\omega_\mu^{(\lambda)}$ $(i=1,2;$ $\mu = 1, 2)$ by

$$q_i(\lambda) = \sum_{n=-\infty}^{\infty} \lambda^n q_i^{(n)} \qquad \omega_\mu^{(\lambda)} = \sum_{n=-\infty}^{\infty} \lambda^n \omega_\mu^{(n)}. \tag{4.8}$$

Then we can unify the infinite number of differential equations (4.4) ~ (4.7) into the following equations

$$\begin{aligned}\partial_\eta q(\lambda) &= \tfrac{1}{2}[(1/\lambda^2)\sigma_3 + \mathrm{i}\pi\sigma_2]q(\lambda) - (\mathrm{i}/\sqrt{2}\lambda)\psi_1\sigma_2\omega(\lambda) \\ \partial_\xi q(\lambda) &= \lambda^2(-\sin\phi\sigma_1 + \cos\phi\sigma_3)q(\lambda) - (\lambda/\sqrt{2})\psi_2[\sin(\phi/2)\sigma_3 + \cos(\phi/2)\sigma_1]\omega(\lambda)\end{aligned} \tag{4.9}$$

$$\begin{aligned}\partial_\eta\omega(\lambda) &= (1/2\lambda^2)\sigma_3\omega(\lambda) - (\mathrm{i}/\sqrt{2}\lambda)\psi_1\sigma_1 q(\lambda) \\ \partial_\xi\omega(\lambda) &= (\lambda^2/2)\sigma_3\omega(\lambda) - (\lambda/\sqrt{2})\psi_2[\mathrm{i}\sin(\phi/2) - \cos(\phi/2)\sigma_2]q(\lambda)\end{aligned} \tag{4.10}$$

where

$$q(\lambda) = \begin{pmatrix} q_1(\lambda) \\ q_2(\lambda) \end{pmatrix} \qquad \omega(\lambda) = \begin{pmatrix} \omega_1(\lambda) \\ \omega_2(\lambda) \end{pmatrix}. \tag{4.11}$$

Equations (4.9) and (4.10) give the Lax pair of the supersymmetric sine-Gordon equation. When we introduce a two-component superfield $\Phi(\eta, \xi, \theta)$, where θ is the Grassmann number $(\theta^2 = 0)$, on the superspace with coordinates $\{\eta, \xi, \theta\}$ by

$$\Phi(\eta, \xi, \theta) = q(\eta, \xi) + \theta\omega(\eta, \xi) \tag{4.12}$$

we can represent (4.9) and (4.10) in the form

$$\begin{aligned}\mathrm{i}\partial_\eta\Phi(\eta, \xi, \theta) &= L\Phi(\eta, \xi, \theta) \\ \mathrm{i}\partial_\xi\Phi(\eta, \xi, \theta) &= K\Phi(\eta, \xi, \theta).\end{aligned} \tag{4.13}$$

In (4.3) L and K are matrix-valued operators in the superspace and are defined by

$$\begin{aligned}L &= L_1 + L_2\partial_\theta + \theta L_3 + \theta L_4\partial_\theta \\ K &= K_1 + K_2\partial_\theta + \theta K_3 + \theta K_4\partial_\theta\end{aligned} \tag{4.14}$$

where L_i and K_i are 2×2 matrices and are given by

$$\begin{aligned} L_1 &= \frac{\mathrm{i}}{2}\left(\frac{1}{\lambda^2}\sigma_3 + \mathrm{i}\pi\sigma_2\right) \qquad & L_2 &= \frac{1}{\sqrt{2}\lambda}\psi_1\sigma_2 \\ L_3 &= \frac{1}{\sqrt{2}\lambda}\psi_1\sigma_2 \qquad & L_1 + L_4 &= \frac{\mathrm{i}}{2\lambda^2}\sigma_3\end{aligned} \tag{4.15}$$

and

$$K_1=\frac{i\lambda^2}{2}(-\sin\phi\sigma_1+\cos\phi\sigma_3) \qquad K_2=-\frac{i\lambda}{\sqrt{2}}\psi_2\left(\sin\frac{\phi}{2}\sigma_3+\cos\frac{\phi}{2}\sigma_1\right)$$
$$K_3=\frac{\lambda}{2}\psi_2\left(\sin\frac{\phi}{2}+i\cos\frac{\phi}{2}\sigma_2\right) \qquad K_1+K_4=\frac{i\lambda^2}{2}\sigma_3. \tag{4.16}$$

We can show that (4.13) is equivalent to the super Lax pair given in our previous paper [12]. Then we can conclude that the differential system of the supersymmetric sine-Gordon equation has the prolongation structure and that there is an infinite-dimensional superalgebra associated with the structure whose representation gives the super Lax pair of the equation.

The scattering problem of the Lax pair for the supersymmetric sine-Gordon equation has been shown to be well defined in the superspace [12, 13]. In the superspace formulation of the scattering problem an S matrix is defined by

$$S=S_1+S_2\partial_\theta+\theta S_3+\theta(S_4-S_1)\partial_\theta \tag{4.17}$$

where $S_i(i=1,2,3,4)$ are 2×2 matrices and are expressed in terms of scattering parameters $a_i(\lambda)$, $b_i(\lambda)$, $c_i(\lambda)$, $d_i(\lambda)$ $(i=1,2,3,4)$ as

$$S_i=\begin{bmatrix} a_i(\lambda), & b_i(\lambda) \\ c_i(\lambda), & d_i(\lambda)\end{bmatrix}. \tag{4.18}$$

The time dependence of the scattering parameters are given by

$$\begin{bmatrix} a_i(t), & b_i(t) \\ c_i(t), & d_i(t)\end{bmatrix}=\begin{bmatrix} a_i(0), & b_i(0)\,e^{-2ik_0t} \\ c_i(0)\,e^{2ik_0t} & d_i(0)\end{bmatrix} \tag{4.19}$$

where $t=\eta-\xi$ and $k_0=\frac{1}{4}(\lambda^2+1/\lambda^2)$. Then we find scattering parameters $a_i(\lambda)(i=1,2,3,4)$ do not change with time. By expanding these parameters into power series of λ as follows:

$$\log a_1(\lambda)=\begin{cases}\sum_{n=0}^{\infty}\lambda^{-2n}H_{-n} & |\lambda|\to\infty \\ \sum_{n=0}^{\infty}\lambda^{2n}H_n & |\lambda|\to 0\end{cases} \tag{4.20}$$

$$\log a_4(\lambda)=\begin{cases}\sum_{n=0}^{\infty}\lambda^{-2n}\Omega_{-n} & |\lambda|\to\infty \\ \sum_{n=0}^{\infty}\lambda^{2n}\Omega_n & |\lambda|\to 0\end{cases} \tag{4.21}$$

$$a_2(\lambda)=\begin{cases}\sum_{n=0}^{\infty}\lambda^{-2n-1}\Lambda_{-n} & |\lambda|\to\infty \\ \sum_{n=0}^{\infty}\lambda^{2n+1}\Lambda_n & |\lambda|\to 0\end{cases} \tag{4.22}$$

we can show that all coefficients of the above expansions constitute sets of an infinite number of conserved quantities.

Some examples of these conserved quantities are given by

$$H = 2\mathrm{i}[(H_1 - \Omega_1) - (H_{-1} - \Omega_{-1})]$$

$$P = -2\mathrm{i}[(H_1 - \Omega_1) + (H_{-1} - \Omega_{-1})]$$

$$Q_1 = 8\sqrt{2}\mathrm{i}\Lambda_{-0} \qquad Q_2 = -8\sqrt{2}\Lambda_0$$

where Q_1, Q_2 are generators of the supertransformations and have odd characters of the Grassmann algebra.

Since it can be shown that other time-independent scattering parameters $a_3(\lambda)$ and d_i $(i = 1, 2, 3, 4)$ are not independent of $a_1(\lambda)$, $a_2(\lambda)$ and $a_4(\lambda)$ we can conclude that all bose-like and fermi-like conserved quantities of the supersymmetric sine-Gordon equation are given by $H_{\pm n}$, $\Omega_{\pm n}$ and $\Lambda_{\pm n}$.

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